Logical Analysis of Ramsey's Theorem

by
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1. Introduction

In 1931 F. P. Ramsey proved the following:

Theorem 1.1. RT^n_k: *If the collection of n-element subsets of the natural numbers $\mathbb{N}$ is partitioned into k colors, then there is an infinite set H all of whose n-element subsets have the same color.*

For an integer $n > 0$, let $[n]$ denote the set $\{i | i \leq n\}$. Let $[n]^r$ denote the r-element subsets of $[n]$. A finite version of RT^n_k may be stated as follows:

Theorem 1.2. *For any positive integers l, k, r, there is an $n > l$ such that if $[n]^r$ is partitioned into k colors, then there is a subset H of $[n]$, with size l, such that all r-element subsets of H have the same color.*

Theorem 1.2 is a finite version of Theorem 1.1 in the sense that $n \to \infty$ as $l \to \infty$. Although at first glance RT^n_k and its finite version appear to be just combinatorial statements about coloring and therefore perhaps of limited interest, mathematical developments over the last several decades have pointed to important links between them and different branches of mathematics. The following lists some of the most notable connections:

(1) *Logic*: The famous incompleteness theorem of Gödel states that if the first order theory of Peano arithmetic (which essentially consists of the standard axioms concerning arithmetic operations, plus mathematical induction) is consistent (i.e. it does not prove statements like $0 \neq 0$. More on this later), then there is a statement about the natural numbers that, while true, cannot be proved within the system. In 1974, Paris and Harrington used a variant of the finite Ramsey Theorem, which is in fact a consequence of the latter, to prove Gödel's theorem. This was the first natural example of an unprovable true statement about the integers.

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(2) *Topological dynamics*: The work of Furstenberg (1981) that proves Szemerédi's Theorem (1975) on arithmetic progression.

(3) *Model theory*: The existence of indiscernibles in models of countable complete theories, an important ingredient in the proof of Morley's celebrated categoricity theorem (1965).

(4) *Geometry of Banach spaces*: The proof of Rosenthal's theorem on l_1 space and Grower's positive solution of the homogeneity problem of Banach spaces, and the subsequent development in the 1980's and 1990's.

We consider here Ramsey's Theorem from the logical point of view. In early 20th century, David Hilbert embarked on the study of foundations of mathematics, identifying as a major program the proof of the consistency of mathematics from a set of axioms. That ambitious project was dealt a death blow by Gödel's Incompleteness Theorem. In the ensuing years, the study of the foundations of mathematics has assumed a heavy philosophical bent, and mathematical logic has moved on to explore uncharted territories, sometimes completely ignoring foundational issues.

The work of Harvey Friedman, Stephen Simpson and others, perhaps beginning with Friedman's MIT doctoral thesis of 1968 and continuing into the 1980's and 1990's, has led to the creation of a new subject called *reverse mathematics* which may be viewed as a new chapter in Hilbert's program.

The central question in reverse mathematics is the following: Which set existence axiom in subsystems of second order arithmetic is necessary, or sufficient, or both, to prove theorems in mathematics? To make this question precise, we begin by recalling some basic notions.

2. Subsystems of second order arithmetic

The Peano axioms consist of the following:

(i) 0 is a number;

(ii) If x is a number, $x+1$ (the successor of x) is a number;

(iii) 0 is not the successor of any number;

(iv) If $x+1=y+1$, then $x=y$;

(v) (Mathematical Induction) If 0 satisfies property Φ, and $x+1$ satisfies property Φ whenever x does, then all numbers satisfy Φ.

(vi) (Comprehension Axiom) For any property Φ about numbers x, the collection $\{x|x \text{ satisfies } \Phi\}$ is a set.

We remark that there is some subtlety in (v) and (vi). Namely, the truth of (v) or (vi) depends on the context in which it is considered. For example, if we only allow computable sets in our world, then non-computable sets are cast out of the picture. In that situation, $\{x|x \text{ satisfies } \Phi\}$ may not be computable and therefore is not a set in the world being considered. The Φ in (v) and (vi) refer to *second order* number-theoretic properties, i.e. statements that may mention subsets of the natural numbers $\mathbb{N}$ (second order) or numbers themselves (first order), but not sets of sets of numbers (third order). Thus a sequence of numbers may be mentioned in Φ (for example, Φ may be the statement: x is a limit point of an infinite sequence A), but not the set of all infinite sequences. (Note that it makes sense to talk about sequences and limits in what apparently is a universe consisting only of natural numbers, since the notions of rational numbers, limits etc. may be coded in any model that satisfies the system RCA_0 which we define below.)

There is a hierarchy of subsystems of second order arithmetic determined by the strength of the set existence axiom in the subsystem. This is defined in terms of the complexity of Φ in (v) and (vi):

First of all, we restrict mathematical induction to what is called "Σ_1^0 statements". A typical example of a Σ_1^0 statement would be $\exists z(x+1=z)$, or $\exists z \forall y \leq x(z > y)$. From the computational point of view, a Σ_1^0 statement is one where a computer program will print an output if and when a z for the $\exists z$ is found, and no output otherwise (there need not be an a priori knowledge of whether a z exists). A statement like $\exists z \forall y(\dots)$ is not Σ_1^0 since in principle no computer program can decide if the z being considered works for all y. Σ_1^0 induction provides a bare minimum for any system to be sufficiently strong to derive nontrivial mathematical results.

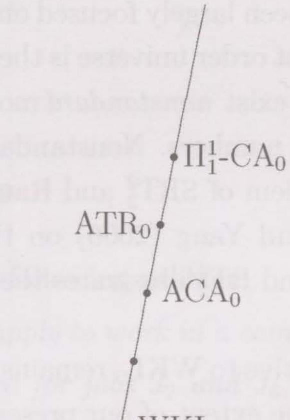
Next, for (vi) we restrict Φ to those that are computable from sets previously admitted into the model being considered. This is called *Recursive Comprehension Axiom.* The basic subsystem of second order arithmetic consists of Σ_1^0 induction plus recursive comprehension. It is denoted RCA_0. A good example of a model for RCA_0 is the structure consisting of the set of natural numbers, plus the collection of all computable sets (as the class of second order objects in the model).

It turns out that RCA_0 is a fairly powerful system. Many of the basic notions of analysis may be formulated within the system. It is known, for example, that RCA_0 is equivalent to the Intermediate Value Theorem in calculus.

A system that is strictly stronger than RCA_0 is Weak König's Lemma WKL_0: Every infinite binary tree has an infinite path. Several classical results in mathematics, such as the Heine-Borel Theorem, Brouwer's Fixed Point Theorem, the local existence theorem for solutions of systems of differential equations and so on have been shown to be equivalent to WKL_0 over RCA_0.

Next up in the hierarchy of subsystems is RCA_0 endowed with *arithmetic comprehension*: Any property that is described using numbers and sets is satisfied by a set within a model of such a system. Thus, for example, if an infinite set X of numbers satisfies the axioms in the system, then many (indeed infinitely many) infinite subsets of X do. This system is denoted ACA_0. It is known that, over RCA_0, ACA_0 is stronger than WKL_0 and equivalent to the Bolzsano-Weierstrass Theorem as well as the classical König's Lemma: Any infinite finitely branching tree has an infinite path.

Beyond ACA_0, there are stronger systems such as ATR_0 (Arithmetic Transfinite Recursion) and Π^1_1-comprehension. We shall not get into the technical details, except to point out that the former is equivalent to the theorem that every uncountable set of reals contains a perfect subset, while the latter is equivalent to the Cantor-Bendixson Theorem stating that every uncountable set of reals is the union of a perfect set and a countable set. The following picture is a summary of the relative strengths of these systems:



The strength of Ramsey's Theorem (Theorem 1.1) in this hierarchy has been the subject of active research in recent years. Surprisingly, there is a striking difference in terms of proof-theoretic strength between RT^2_2 and RT^n_2 for $n > 2$ (it is straightforward to verify that RT^n_k is equivalent to RT^n_2 for any k, n).

3. The strength of Ramsey's Theorem

An earlier work of Jockusch (1972) implies that RT^n_2 is equivalent to ACA_0. For a long time it was not clear where RT^2_2 stood. A major result of Seetapun and Slaman in 1995 shows that RT^2_2 is in fact strictly weaker than RT^n_2 for $n > 2$. Although more expressive power becomes available when considering exponent $n > 2$, the exact reason for the relative strength of RT^n_2 over RT^2_2 came somewhat as a surprise.

There is a refinement of the problem of 2-coloring of pairs. If f is a partition of $\mathbb{N}^2$ into two colors, then we say that f is *stable* if for all x, $\lim_s f(x,s)$ exists. Denote by SRT^2_2 the statement that every stable 2-coloring of pairs has an infinite set all of whose 2-element subsets have the same color. Clearly RT^2_2 implies SRT^2_2. The conjecture is that the converse is false. Despite many attempts, it remains open.

From the point of view of computability theory, the conjecture is entirely reasonable. Jockusch (1972) proved that in general, a solution for a 2-coloring of pairs can be fairly complicated. In fact he exhibited a 2-coloring where no solution is computable in the halting problem of Turing. By contrast, any stable 2-coloring admits a solution that is computable in the halting problem. Thus, to establish the conjecture, it is necessary to construct a model of $RCA_0 + SRT^2_2$ in which RT^2_2 fails. The most obvious approach is to ensure that Jockusch's 2-coloring has no solution in the model. It is not clear how such a model can be obtained.

Traditionally, work in RT^n_k has been largely focused on studying the class of "ω-models", mathematical structures whose first order universe is the set of natural numbers. However, in any system of arithmetic, there exist *nonstandard* models whose universe consist of the natural numbers *and* nonstandard numbers. Nonstandard models of $RCA_0 + RT^2_2$ provide a different perpective to the problem of SRT^2_2 and Ramsey type problems in general. A recent work of Chong, Slaman and Yang (2006) on the strength of the combinatorial principle COH ("cohesiveness") and RT^2_2 illustrates the relevance of nonstandard models in such investigations.

Finally, the strength of RT^2_2 relative to WKL_0 remains unsolved. It is known that WKL_0 does not prove RT^2_2, but that is the extent of our present knowledge. Interestingly, while arguments concerning RT^2_2 make essential use of the existence of infinite paths on binary trees, it is not obvious how this use can be formalized and turned into a proof that RT^2_2 implies WKL_0. The problem appears to be difficult.

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